A KINEMATICAL PROPERTY OF RULED SURFACES.*

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Let S be any ruled surface, not developable, with real rulings, C a real curve of S, not a straight line, intersecting all rulings considered, and g any such ruling; let the coördinates of a point of C be x_0, y_0, z_0; suppose C rectifiable and let its arc be v; suppose C has at every point a definite principal trihedral, and let α, β, γ be the direction cosines of the tangent, l, m, n those of the principal normal, λ, μ, ν those of the binormal; let R and T be the radii of curvature and torsion respectively; let ψ be the angle made by the direction chosen as positive on g with the osculating plane of C measured towards the positive binormal, φ the angle of the plane of g and the binormal measured from the rectifying plane towards the normal plane. (If g coincides with the binormal φ is not determined.) The surface S is given by

$$(1)\qquad x = x_0 + uL, \qquad L = \alpha \cos\psi \cos\varphi + l \cos\psi \sin\varphi + \lambda \sin\psi,$$

with similar equations for y and z, where u is the length measured on g from C in the positive direction. Supposing ψ and φ to have finite first derivatives with respect to v the linear element of S is given by

$$(2)\qquad \begin{aligned} ds^2 = (du + \cos\psi \cos\varphi dv)^2 + \{&\cos^2\psi \sin^2\varphi + \sin^2\psi - 2u[\sin\psi \cos\varphi\psi' \\ &+ \cos\psi \sin\varphi(\varphi' + 1/R)] + u^2[((\sin\varphi)/T - \psi')^2 \\ &+ (\cos\psi\{\varphi' + 1/R\} + (\sin\psi \cos\varphi)/R)^2]\}dv^2. \end{aligned}$$

The coefficient of u^2 in (2) does not vanish identically since S is not developable.† The linear element of a ruled surface may be given the form‡

$$(3)\qquad ds^2 = du_1^2 + [(u_1 - a)^2 + b^2]dv_1^2,$$

where the curves (v_1), that is v_1 constant, are the rulings, u_1 is the length measured on a ruling from the orthogonal trajectory, $u_1 = 0$; the function a, depending on v_1 alone, is the distance from this trajectory to the central point of the ruling (v_1), so that the equation of the line of striction σ is $u_1 = a$, and b also depending on v_1 alone is the parameter of distribution.

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* Presented to the American Mathematical Society, April 28, 1917.

† Darboux, Théorie des surfaces, vol. 3, p. 294.

‡ Darboux, vol. 1, 2d ed., p. 122.

Comparing (2) and (3) it is evident that the necessary and sufficient condition that C be σ is

$$\sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R) = 0. \tag{4}$$

The parameter of distribution of S is*

$$b = -\,|\,x_0'MN'\,|\,/\Sigma L'^2, \tag{5}$$

where L, M, N are the coefficients of u in (1), and the numerator of the second member is a determinant of which we have written the principal diagonal. If C is σ and is not a geodesic of S, so that $\varphi \neq 0$, (5) gives, using (4),

$$\frac{1}{b} = \frac{1}{T} - \frac{\psi'}{\sin\varphi}. \tag{6}$$

Supposing now that C is the line of striction σ of a ruled surface S, and that C is not a geodesic of S, let ξ, η, ζ be the coördinates of any point of the ruling g referred to the principal trihedral of C, and referred also to axes fixed in space and coincident with the principal trihedral; let d and δ be differentials referring to the moving and fixed axes respectively. We have†

$$\frac{\delta\xi}{dv} = \frac{d\xi}{dv} + 1 - \frac{\eta}{R}, \qquad \frac{\delta\eta}{dv} = \frac{d\eta}{dv} + \frac{\xi}{R} + \frac{\zeta}{T}, \qquad \frac{\delta\zeta}{dv} = \frac{d\zeta}{dv} - \frac{\eta}{T},$$

$$\xi = u\cos\psi\cos\varphi, \qquad \eta = u\cos\psi\sin\varphi, \qquad \zeta = u\sin\psi.$$

From these equations

$$\frac{\delta\xi}{dv} = 1 - u[\sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R)],$$

$$\frac{\delta\eta}{dv} = u[\sin\psi(1/T - \sin\varphi\psi') + \cos\psi\cos\varphi(\varphi' + 1/R)],$$

$$\frac{\delta\zeta}{dv} = -\,u\cos\psi((\sin\varphi)/T - \psi').$$

We find from the preceding equations

$$\eta\frac{\delta\eta}{dv} + \zeta\frac{\delta\zeta}{dv} = u^2\cos\psi\cos\varphi[\sin\psi\cos\varphi\psi' + \cos\psi\sin\varphi(\varphi' + 1/R)],$$

$$\zeta\frac{\delta\eta}{dv} - \eta\frac{\delta\zeta}{dv} = u^2\left[\frac{\cos^2\psi\sin^2\varphi + \sin^2\psi}{T} - \sin\varphi\psi'\right.$$

* Darboux, vol. 3, p. 302.

† Eisenhart, Differential Geometry, p. 32.

$$+ \sin\psi \cos\psi \cos\varphi\left(\varphi' + \frac{1}{R}\right)\Big] = u^2\Big[(\cos^2\psi \sin^2\varphi$$

$$+ \sin^2\psi)\left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right) + \frac{\sin\psi\cos\psi}{\sin\varphi}\Big(\sin\psi\cos\varphi\psi'$$

$$+ \cos\psi \sin\varphi\left\{\varphi' + \frac{1}{R}\right\}\Big)\Big].$$

Using (4) and (6) these give*

$$\frac{\delta\xi}{dv} = 1, \qquad \eta\frac{\delta\eta}{dv} + \zeta\frac{\delta\zeta}{dv} = 0, \qquad \zeta\frac{\delta\eta}{dv} - \eta\frac{\delta\zeta}{dv} = \frac{\eta^2 + \zeta^2}{b}. \tag{7}$$

Equations (7) hold in the case, excluded from the preceding discussion, where σ is a curved geodesic of S, and also with the proper modifications in the definitions of the coördinates where σ is a straight line.

From (7) follows

THEOREM I. *Every ruled surface S may be generated by a radius fixed in a sphere whose center moves with unit velocity along the line of striction and which turns about the tangent to this curve from the binormal towards the principal normal with angular velocity equal to the reciprocal of the parameter of distribution.*

There follow two corollaries:

If two ruled surfaces have the same line of striction and equal parameters of distribution their rulings intersect on the line of striction at a constant angle.

If a sphere moves so that it always rotates about the tangent to the locus of its center every radius fixed in the sphere generates a ruled surface whose line of striction is the locas of the center; all such surfaces have the same parameter of distribution.

From (6), if σ is not a geodesic of S, follows

THEOREM II. *A necessary and sufficient condition that the parameter of distribution of a ruled surface S be equal to the radius of torsion of the line of striction at the corresponding point is that the ruling form a constant angle with the binormal of the line of striction at the point of intersection.*

The case where the line of striction is a geodesic is included in Theorem II, for it may be shown that in this case the parameter of distribution is equal to the radius of torsion of the line of striction when and only when the ruling coincides with the binormal.

From (4) it follows if $\psi' = 0$, $\sin\varphi \neq 0$ and S is not a binormal surface (*i. e.*, $\cos\psi \neq 0$), that for all ruled surfaces S, such that $b = T$ on σ, except binormal surfaces,

$$\psi' = 0, \qquad \varphi' + 1/R = 0. \tag{8}$$

Such a surface we call a T surface. From Theorem I it follows that the

* Equations (7) follow from equations given by Cesàro, Geometria intrinseca, pp. 134, 135. Cesàro makes no application similar to that here given.

osculating plane of σ is fixed in the generating sphere of S, and conversely, that if the osculating plane of σ is fixed in the sphere S is a T surface or a binormal surface. A T surface is uniquely determined by choosing any curve C as σ, and any line intersecting C, not a binormal, as a ruling.

We consider deformations connected with T surfaces, proving that any ruled surface S, on which σ is not a geodesic, is applicable with correspondence of rulings to a T surface, and that a T surface, if $\psi \neq 0$, may be continuously deformed while remaining a T surface. For any surface S, on which σ is not a geodesic, we have from (2) and (4)

$$ds^2 = (du + \cos\psi \cos\varphi dv)^2 + (\cos^2\psi \sin^2\varphi + \sin^2\psi)\left[1 + u^2\left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right)^2\right] dv^2.$$

Using subscripts for a T surface S_1,

$$ds_1^2 = (du + \cos\psi_1 \cos\varphi_1 dv)^2 + (\cos^2\psi_1 \sin^2\varphi_1 + \sin^2\psi_1)\left(1 + \frac{u^2}{T_1^2}\right) dv^2.$$

For applicability of S and S_1 the lines of striction must correspond. Necessary and sufficient conditions for applicability are

$$\text{(9)} \qquad \cos\psi_1 \cos\varphi_1 = \cos\psi \cos\varphi, \qquad \frac{1}{T_1^2} = \left(\frac{1}{T} - \frac{\psi'}{\sin\varphi}\right)^2,$$

with $\psi_1' = \varphi_1' + 1/R_1 = 0$, and (4) holding for S. By differentiating the first of (9), we find on reduction by means of (4)

$$\frac{\cos\psi_1 \sin\varphi_1}{R_1} = \frac{\cos\psi \sin\varphi}{R}.$$

If S is given, S_1 may be determined to satisfy these conditions as follows: ψ_1 may be chosen arbitrarily subject to the condition, $|\cos\varphi_1| = |\cos\psi \cos\varphi \sec\psi_1| \leqq 1$. If we consider the deformation of a given part of S it is clear that the numerical value of ψ_1 is at most equal to the smallest numerical value of the angle of the ruling and the tangent to σ on that part of S considered; if this value is zero the only possible choice is $\psi_1 = 0$, and σ_1 is an asymptotic line of S_1. When ψ_1 is chosen, $\cos\varphi_1$ is determined, then the sign of $\sin\varphi_1$, hence φ_1 and the positive R_1. The torsion of σ_1 is determined except as to sign, but if the deformation of S into S_1 is continuous this sign is also fixed. Then σ_1 and φ_1 and consequently S_1 are determined by the choice of ψ_1 except for the sign of T_1 in non-continuous deformation. The statements made concerning the continuous deformation of a T surface while remaining a T surface follow from the preceding discussion. It is obvious that the lines of striction of all T surfaces applicable to a given surface S have the same torsion numerically at corresponding points. If the parameter of distribution of S is constant the line of striction of an applicable T surface is a curve of constant torsion.

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